A POLAR DECOMPOSITION BASED DISPLACEMENT METRIC FOR A FINITE REGION OF SE(N)

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Abstract An open research question is how to define a useful metric on SE(n) with respect to (1) the choice of coordinate frames and (2) the units used to measure linear and angular distances. A technique is presented for approximating elements of the special Euclidean group SE(n) with elements of the special orthogonal group SO(n+1). This technique is based on the polar decomposition (denoted as PD) of the homogeneous transform representation of the elements of SE(n). The embedding of the elements of SE(n) into SO(n+1) yields hyperdimensional rotations that approximate the rigid-body displacement. The bi-invariant metric on SO(n+1) is then used to measure the *distance* between any two spatial displacements. The result is a PD based metric on SE(n) that is left invariant. Such metrics have applications in motion synthesis, robot calibration, motion interpolation, and hybrid robot control.

Keywords: Displacement metrics, metrics on the special Euclidean group, rigid-body displacements

1. Introduction

Simply stated a metric measures the distance between two points in a set. There exist numerous useful metrics for defining the distance between two points in Euclidean space, however, defining similar metrics for determining the distance between two locations of a finite rigid body is still an area of ongoing research, see Kazerounian and Rastegar, 1992, Martinez and Duffy, 1995, Larochelle and McCarthy, 1995, Etzel and McCarthy, 1996, Gupta, 1997, Tse and Larochelle, 2000, Chirikjian, 1998, Belta and Kumar, 2002, and Eberharter and Ravani, 2004. In the cases of two locations of a finite rigid body in either SE(3) (spatial locations) or SE(2) (planar locations) any metric used to measure the distance between the locations yields a result which depends upon the chosen reference frames, see Bobrow and Park, 1995 and Martinez and Duffy, 1995. However, a metric that is independent of these choices,

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referred to as being bi-invariant, is desirable. Interestingly, for the specific case of orienting a finite rigid body in SO(n) bi-invariant metrics do exist.

Larochelle and McCarthy, 1995 presented an algorithm for approximating displacements in SE(2) with spherical orientations in SO(3). By utilizing the bi-invariant metric of Ravani and Roth, 1983 they arrived at an approximate bi-invariant metric for planar locations in which the error induced by the spherical approximation is of the order $\frac{1}{R^2}$, where R is the radius of the approximating sphere. Their algorithm for an approximately bi-invariant metric is based upon an algebraic formulation which utilizes Taylor series expansions of *sine*() and *cosine*() terms in homogeneous transforms, see McCarthy, 1983. Etzel and McCarthy, 1996 extended this work to spatial displacements by using orientations in $SO(4)$ to approximate locations in $SE(3)$. This paper presents an alternative approach for defining a metric on SE(n). Here, the underlying geometrical motivations are the same- to approximate displacements with hyperspherical rotations. However, an alternative approach for reaching the same goal is presented. The polar decomposition is utilized to yield hyperspherical orientations that approximate planar and spatial finite displacements.

2. The PD Based Embedding

This approach, analogous to the works reviewed above, also uses hyperdimensional rotations to approximate displacements. However, this technique uses products derived from the singular value decomposition (SVD) of the homogeneous transform to realize the embedding of SE(n-1) into SO(n). The general approach here is based upon preliminary work reported in Larochelle et al., 2004.

Consider the space of $(n \times n)$ matrices as shown in Fig. 1. Let $[T]$ be a $(n \times n)$ homogeneous transform that represents an element of SE(n-1). $[A]$ is the desired element of SO(n) nearest $[T]$ when it lies in a direction orthogonal to the tangent plane of SO(n) at $[A]$. The PD of $[T]$ is used to determine $[A]$ by the following methodology.

The following theorem, based upon related works by Hanson and Norris, 1981 provides the foundation for the embedding

Theorem 1. *Given any $(n \times n)$ matrix $[T]$ the closest element of SO(n) is given by: $[A] = [U][V]^T$ where $[T] = [U][diag(s_1, s_2, \ldots, s_n)][V]^T$ is the SVD of $[T]$.*

Shoemake and Duff, 1992 prove that matrix $[A]$ satisfies the following optimization problem: *Minimize:* $\|[A]-[T]\|_F^2$ subject to: $[A]^T[A]-[I] = [0]$, where $\|[A]-[T]\|_F^2 = \sum_{i,j}(a_{ij}-t_{ij})^2$ is used to denote the Frobenius

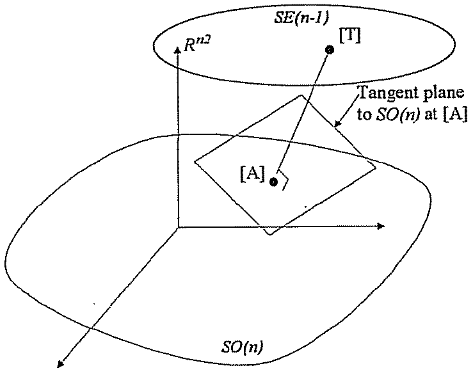

Figure 1. General Case: SE(n-1) ⇒ SO(n).

norm. Since $[A]$ minimizes the Frobenius norm in R^{n^2} it is the element of SO(n) that lies in a direction orthogonal to the tangent plane of SO(n) at $[R]$. Hence, $[A]$ is the closest element of SO(n) to $[T]$. Moreover, for full rank matrices the SVD is well defined and unique. Th. 1 is now restated with respect to the desired SVD based embedding of SE(n-1) into SO(n).

Theorem 2. *For $[T] \in SE(n\text{-}1)$ and $[U]$ & $[V]$ are elements of the SVD of $[T]$ such that $[T] = [U][diag(s_1, s_2, \ldots, s_{n-1})][V]^T$ if $[A] = [U][V]^T$ then $[A]$ is the unique element of $SO(n)$ nearest $[T]$.*

Recall that $[T]$, the homogenous representation of SE(n), is full rank (McCarthy, 1990) and therefore $[A]$ exists, is well defined, and unique.

The polar decomposition is quite powerful and actually provides the foundation for the better known singular value decomposition. The polar decomposition theorem of Cauchy states that "a non-singular matrix equals an orthogonal matrix either pre or post multiplied by a positive definite symmetric matrix", see Halmos, 1958. With respect to our application, for $[T] \in$ SE(n-1) its PD is $[T] = [P][Q]$, where $[P]$ and $[Q]$ are $(n \times n)$ matrices such that $[P]$ is orthogonal and $[Q]$ is positive definite and symmetric. Recalling the properties of the SVD, the decomposition of $[T]$ yields $[U][\text{diag}(s_1, s_2, \ldots, s_{n-1})][V]^T$, where matrices $[U]$ and $[V]$ are orthogonal and matrix $[\text{diag}(s_1, s_2, \ldots, s_{n-1})]$ is positive definite and symmetric. Moreover, it is known that for full rank square matrices that the polar decomposition and the singular value decomposition are related by: $[P] = [U][V]^T$ and $[Q] = [V][\text{diag}(s_1, s_2, \ldots, s_{n-1})][V]^T$, Faddeeva,

1959. Hence, for $[A] = [U][V]^T$ it is known that $[A] = [P]$ and the PD yields the same element of SO(n). The result being the following theorem that serves as the basis for the PD based embedding.

Theorem 3. *If $[T] \in SE(n\text{-}1)$ and $[P]$ & $[Q]$ are the PD of $[T]$ such that $[T] = [P][Q]$ then $[P]$ is the unique element of $SO(n)$ nearest $[T]$.*

2.1 The Characteristic Length & Metric

A characteristic length is employed to resolve the unit disparity between translations and rotations. Investigations on characteristic lengths appear in Angeles, 2005; Etzel and McCarthy, 1996; Larochelle and McCarthy, 1995; Kazerounian and Rastegar, 1992; Martinez and Duffy, 1995. The characteristic length used here is $\mathsf{R} = \frac{24L}{\pi}$ where L is the maximum translational component in the set of displacements at hand. This characteristic length is the radius of the hypersphere that approximates the translational terms by angular displacements that are ≤ 7.5(deg). It was shown in Larochelle, 1999 that this radius yields an effective balance between translational and rotational displacement terms. Note that the metric presented here is not dependent upon this particular choice of characteristic length.

It is important to recall that the PD based embedding of SE(n-1) into SO(n) is coordinate frame and unit dependent. However that this methodology embeds SE(n-1) into SO(n) and that a bi-invariant metric does exist on SO(n). One useful metric d on SO(n) can be defined using the Frobenius norm as,

$$d = \|[I] - [A_2][A_1]^T\|_F. \tag{1}$$

where $[A_1]$ and $[A_2]$ of elements of SO(n). It is straightforward to verify that this is a valid bi-invariant metric on SO(n), see Schilling and Lee, 1988.

2.2 A Finite Region of SE(3)

In order to yield a left invariant metric we build upon the work of Kazerounian and Rastegar, 1992 in which approximately bi-invariant metrics were defined for a prescribed finite rigid body. Here, to avoid cumbersome volume integrals over the body a unit point mass model for the moving body is used. Proceed by determining the center of mass $\vec{c}$ and the principal axes frame [PF] associated with the n prescribed locations where a unit point mass is located at the origin of each location:

$$\vec{c} = \frac{1}{n}\sum_{i=1}^{n}\vec{d_i} \tag{2}$$

where $\vec{d_i}$ is the translation vector associated with the i^{th} location (i.e. the origin of the i^{th} location with respect to the fixed frame). Next, define [PF] with origin at $\vec{c}$ and axes along the principal axes of the n point mass system by evaluating the inertia tensor [I] associated with the n point masses,

$$[\mathrm{PF}] = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{c} \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3}$$

where $\vec{v_i}$ are the principal axes associated with [I] Greenwood, 2003 and the directions $\vec{v_i}$ are chosen such that [PF] is a right-handed system. Note that the principal frame is not dependent on the orientations of the frames at hand. However, the metric is dependent on the orientations of the frames. For a set of n locations in a finite region of SE(3) the procedure is:

1 Determine [PF] associated with the n displacements.

2 Determine the relative displacements from [PF] to each of the n locations.

3 Determine the characteristic length R associated with the n relative displacements and scale the translation terms in each by $\frac{1}{\mathsf{R}}$.

4 Compute the elements of SO(4) associated with [PF] and each of the scaled relative displacements using the polar decomposition.

5 The magnitude of the i^{th} displacement is defined as the distance from [PF] to the i^{th} scaled relative displacement as computed via Eq. 1. The distance between any 2 of the n locations is similarly computed via the application of Eq. 1 to the scaled relative displacements embedded in SO(4).

Since $\vec{c}$ and [PF] are invariant with respect to both the choice of coordinate frames as well as the system of units (Greenwood, 2003) the relative displacements determined in step 2 are left invariant and it follows that the metric is also left invariant.

3. Case Study

Consider the 4 spatial locations in Table. 1 and shown in Fig. 2 along with the fixed reference frame [F] where the x-axes are shown in red, the y-axes in green, and the z-axes in blue. Their centroid is $\vec{c} = [0.7500\ 1.5000\ 0.4375]^T$. Next, the principal axes directions are

Table 1. Four Spatial Locations.

#	x	y	z	θ (deg)	ϕ (deg)	ψ (deg)	$\|[T]\|$
1	0.00	0.00	0.00	0.0	0.0	0.0	2.5281
2	0.00	1.00	0.25	15.0	15.0	0.0	2.5701
3	1.00	2.00	0.50	45.0	60.0	0.0	2.7953
4	2.00	3.00	1.00	45.0	80.0	0.0	2.8057

determined to define the principal frame,

$$[\mathrm{PF}] = \begin{bmatrix} -0.5692 & 0.8061 & -0.1617 & 0.75000 \\ -0.7807 & -0.5916 & -0.2012 & 1.5000 \\ -0.2578 & 0.0117 & 0.9661 & 0.4375 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4}$$

shown in Fig. 2. The characteristic length is $\mathrm{R} = \frac{24\times1.7108}{\pi} = 13.0695$ and the magnitudes of the displacements are listed in Table 1. Interestingly, the magnitude of the first displacement is not zero. This is because the relative displacement from the principal frame to the first location is non-identity and that the magnitudes of all displacements are computed with respect to the principal frame.

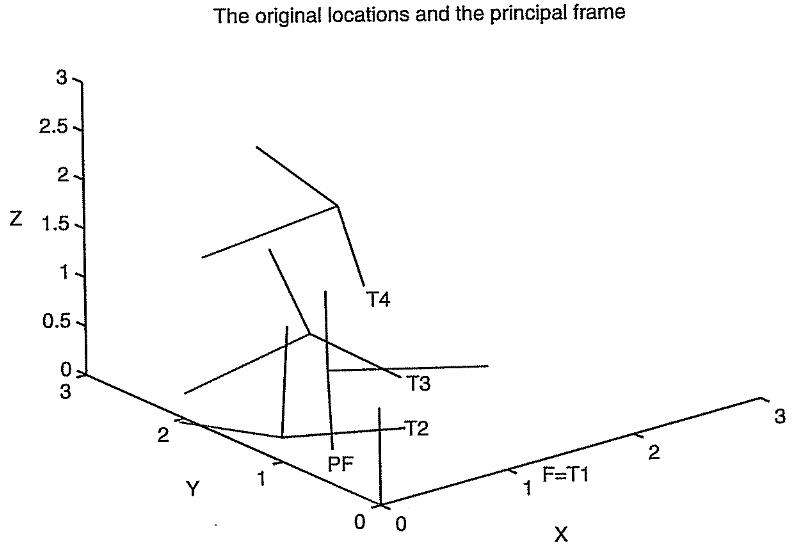

Figure 2. The 4 Spatial Locations.

4. Conclusions

We have presented a metric on SE(n). This metric is based on embedding SE(n) into SO(n+1) via the polar decomposition of the homogeneous transform representation of SE(n). It was shown that this method determines the element of SO(n+1) nearest the given element of SE(n). A bi-invariant metric on SO(n+1) is then used to measure the *distance* between any two spatial displacements SE(n). The results is a PD based metric on SE(n) that is left-invariant. Such metrics have applications in motion synthesis, robot calibration, motion interpolation, and hybrid robot control.

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